The Importance of Visualization in Calculus - The Lemon Problem

Parth Shah, Education 130, Spring 2018

Abstract

I've always been fascinated by the visualization of mathematics. Thus, I created a lesson that allowed for creativity through visualizations. Through cutting lemons and visualizing three dimensional graphs, students were able to learn how to do disk method with minimal help from me. After watching myself teach, I was fascinated by how students used visualizations to address the lemon problem, and I wondered: how do students visualize three dimensional problems? After observing the data of my students, I found that through diverse visualizations, students are able to independently and creatively solve three dimensional problems.

Introduction

In Education 130, I became fascinated by the importance of well designed problems. However, I wanted to go beyond the class and challenge myself to create my own problem. I love the subject of calculus, but it upsets me how it is taught in many classrooms with the prominence of explanations and the lack of self-learning. In the following problem that I present, I demonstrate how visuals and an emphasis on student learning can be used for students to learn disk method.

In my lesson, I had students find the volume of a lemon with only a ruler, a knife, and a lemon. I then showed a demonstration of a rotating graph that formed the volume of a lemon. Through these visualizations, students were able to learn disk method through collaborative efforts. I assumed that the students had a basic knowledge of geometry and integrals and were comfortable working in groups. I was inspired by Jo Boaler's book, Mathematical Mindsets to use visualization and "low-floor high-ceiling problems". Professor Boaler states that these problems are important because they create equity in the classroom. These problems allow students of all backgrounds to approach the problem.

For this paper, I wanted to highlight the importance of visualizations on the cognitive process, specifically how do students visualize three dimensional problems, and what factors do these visualizations play in approaching the problem.

Design

Problem Analysis

Problem Inspiration

Math is a visual subject, and often times, the visualization of mathematics gets lost especially in higher mathematics like calculus. Thus, when I designed this problem, I tried to be as visual as possible. I was inspired by lesson plans in the book, *Mathematical Mindsets* by Jo Boaler, a mathematics education professor at Stanford University. Professor Boaler preaches the idea of "low-floor high ceiling" problems in which a problem is approachable by students of all levels, but is nevertheless challenging in the sense that students can take the problems in many different, creative directions.

Problem Context and Overview

In calculus, students learn volumes of revolution in that they are supposed to learn how to calculate and interpret the volume of shape formed by rotating a graph about an axis.

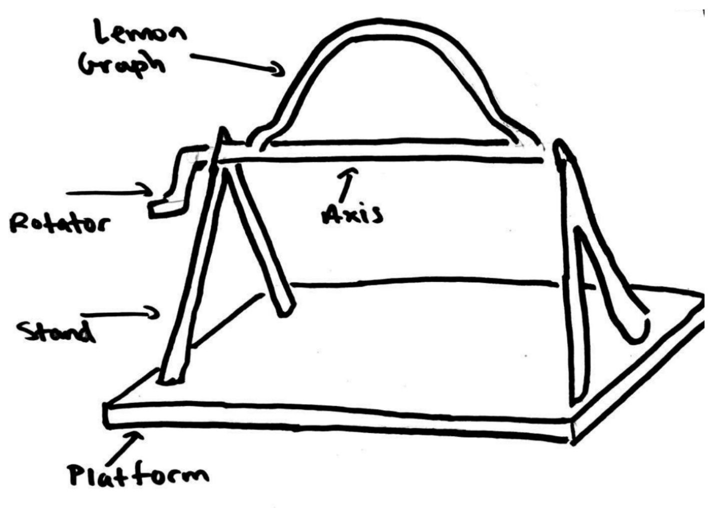
This can seem extremely daunting, and thus I have divided the problem into two parts. In the first part, students will find the volume of a lemon, and in the second part, students will come up with an analytical volume that can apply to any volume of revolution.

Problem Part 1:

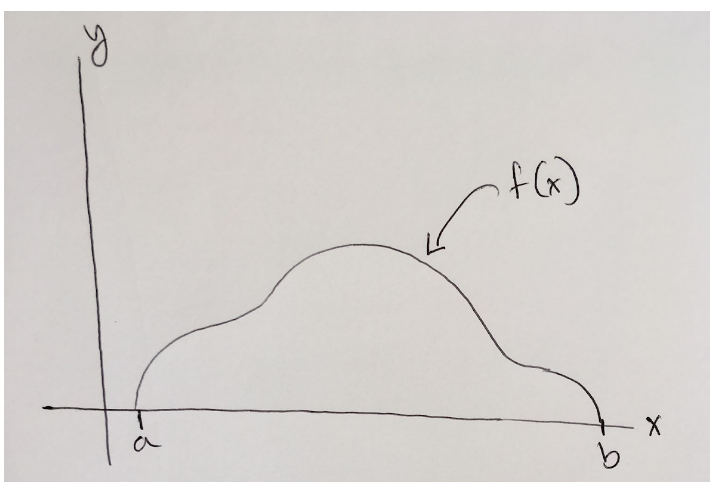
In the first part of this problem, groups of 3-4 will be given a lemon, a knife, a ruler, and a paper towel. In groups, they will be tasked with finding the **BEST ESTIMATE** of the volume of the lemon using any method they see fit. The only hint they will receive is "a lemon is not a sphere". This hint will deter any student from simply finding a radius and using the spherical volume formula. Once a group has an estimate, one group member will write the estimate on the board. After groups have all put up an estimate on the board, one member from each group will speak about how they came up with the volume estimate by sharing their methods.

Problem Part 2:

In the second part of this problem, I will first show the class this model that I built.



This model shows the outline of a lemon on an axis. I will then use the rotator to revolve the lemon about the axis. I will ask students, "What shape is being formed?". They will answer that the shape is a lemon. I will then ask the students, that if the graph of the lemon is *f(x)* and it stretches on the interval (a, b), in groups, write an expression that evaluates to the exact volume of the lemon.



After groups have come up with an expression, they will write their expression on the board. After all groups have came up with an expression, I will ask groups to share how they came up with their expression.

Personal Experience

In high school, I loved calculus. I would wake up at 5 am most days so that I could tutor my fellow classmates at 6 am at the local Starbucks before school started. My senior year, the calculus teacher had to take a leave of absence in the middle of the year, and my principal asked me to drop AP Statistics in order to teach the calculus period. I remember students getting lost at volume of revolutions because it was hard for them to visualize. Thus, when I created this problem, I wanted to add as many visuals as possible so that students could understand volumes at a more conceptual level. I remember that it was particularly difficult for me and my classmates to visualize depth in a graph. Everything in calculus up to that point was in only two dimensions, and that is why we worked in two dimensions. Thus, when we move onto three dimensions in calculus, it seems intuitive to teach in three dimensions, yet the volume unit of calculus is most often taught on paper instead of with three dimensional objects. Therefore, I wanted to teach with three dimensions in order to have my students reach a conceptual understanding of volume.

Prerequisite Knowledge

My favorite aspect of this problem is how approachable it is. As mentioned earlier, I classify this problem as a "low-floor high-ceiling" problem. For the first part of this problem, the only thing students need to know is a basic middle school geometry level of understanding of volume. From this, students can take the problem in many different directions. For the second part of the problem, students can use any skills that they want, whether it be using summations or integrals. Again, students only need a basic understanding of the two to approach the second part of the problem.

Problem Part 1 Pathways Overview

Because of how open-ended the first part of the problem is, there are many different directions that students can take it in. This problem analysis will evaluate **One-Sided Riemann Sum Method**, **Average Riemann Sum Method**, and **Common Shapes Method**.

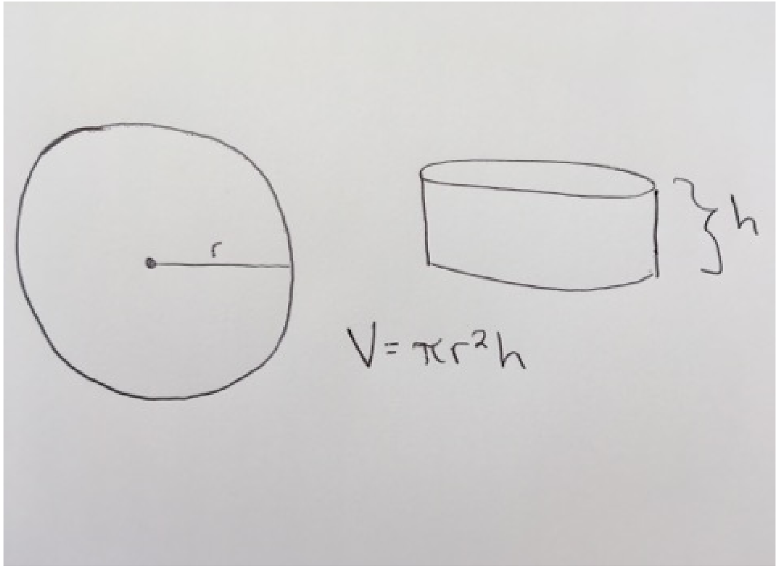
Problem Part 1: One-Sided Riemann Sum Method

In this method, students cut the lemon into multiple cylinders or disks.

Aerial View of Disks

Profile View of Disks

For each disk, the students will then find the radius of either side of the disk and also the height of the disk.



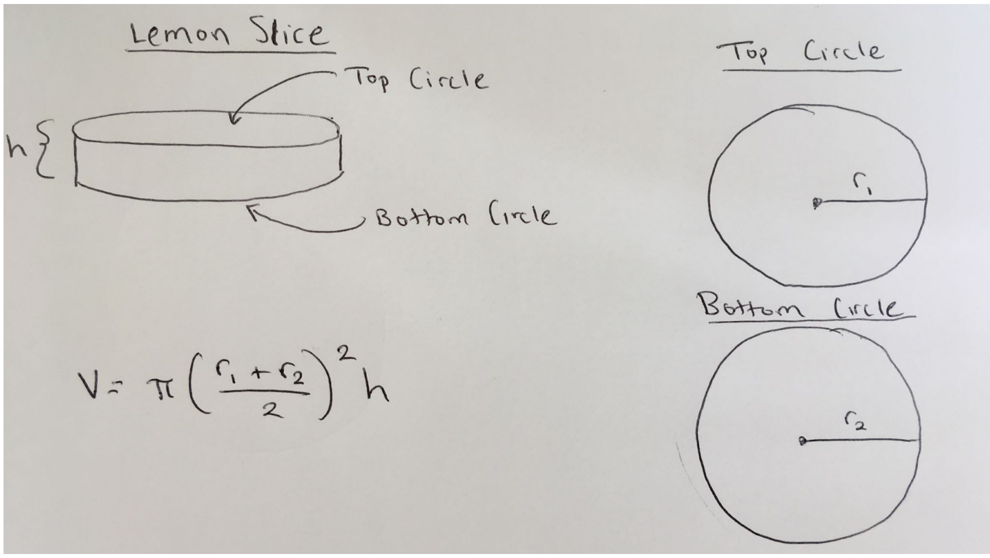
The students will then calculate the volume of each disk by using the volume of a cylinder formula. After finding each volume, they will sum all of the volumes together.

$$\sum_{i=1}^{n} \pi r_i^2 h_i$$

This summation will create an estimate for the volume.

Problem Part 1: Average Riemann Sum Method

This method is extremely similar to the previous method, except that in this method, students will find the radii of both the top and bottom circles of the disks. Students will then average the two radii and then use that average for calculating the volume of each disk.



Students will then sum all the volumes together like in the previous method.

Problem Part 1: Common Shapes Method

In this method, students divide the lemon into common shapes, such as two semi-spheres and a cylinder.

Students then use the formulas to calculate the volume of these familiar shapes to estimate the volume. For instance if r_1 is the radius of the sphere and r_2 is the radius of the cylinder, then students would calculate the volume as:

$$V = \tfrac{4}{3}\pi r_1^{\,3} + \pi r_2^{\,2} h$$

Problem Part 2 Pathways Overview

Like the first part of the problem, in the second part of the problem where students come up with an analytical expression for the volume of the lemon, there are many directions where the students can take it in. This problem analysis will evaluate the **Summation Method**, **Single Integral Method**, **Multiple Integral Method**.

Problem Part 2: Summation Method

In the first part, students saw the volume of lemons as a summation of disks. If the disks were infinitesimally small, then students can use that summation as an exact volume. If the disks are this small, then there is no height, just an infinite sum of circles from the interval (a,b) where each radius is the function evaluated at that point. Thus, students may come up with this expression:

$$\sum_{i=a}^{b} \pi * f(i)^2$$

Note, this answer is not incorrect. However, in calculus, an infinite sum is just an integral. Thus, I may guide students toward an integral if they decide to go towards this route in the problem.

Problem Part 2: Single Integral Method

Like the summation method, in this method, the students see the volume as a sum of disks. They will realize that an infinite sum of disks is just an integral of disks, and they will produce the following expression:

$$\int_a^b \pi * f(x)^2 \, dx$$

This method is formally known as **Disk Method**, a standard in AP Calculus.

Problem Part 2: Multiple Integral Method

Students understand that one integral will give the area of a function, and thus, some may think that two integrals will give the volume. They are not wrong, for they are doing multivariable calculus without realizing it. Thus, they make create an expression that looks like this.

$$\int_0^{2\pi} \int_a^b f(x) \, dx \, d\theta$$

This is not wrong. In fact, this is extremely impressive, as it is multivariable calculus. Students should share this method.

Problem Extensions

If groups find calculating the volume of the lemon easy, I will ask them to try using a different method to find the same volume, or I will ask them to find volume of the lemon peel (Washer's Method).

Connections to Real-World

Students will bring their visualization skills to this problem. They will draw out various ways to attack the problem. They will also bring their organization skills to the first part of the problem in which they may keep a table in order to track all of the variables for the different disks.

Lesson Plan

Objective:

Apply definite integrals to problems involving volume, specifically disk method

Materials:

- Lemons
- Rulers
- Knives
- Paper Towels
- Projector
- Disk Method Simulator
- Index Cards

Set-up:

Put a lemon, a knife, a paper towel, and a ruler on each table. Have the simulator ready for later in the lesson.

Procedure:

Activity 1: Cutting Lemons (25 Min)

The students will enter class and sit in their table groups. At each table, there will be a crate containing a lemon, a ruler, a lab knife, and a paper towel. Instruct students to attempt to find the best estimate volume of the lemon with the given materials. Then, circle the room and talk with groups to learn what their process is. After 20 minutes, have each group facilitator share the process of their respective groups.

5E Connection

The beginning of the activity corresponds to the ***Engage*** part, for students are physically and critically interacting with an object before learning new material. Furthermore, they are ***Exploring*** thought processes as they work in groups and use collaboration phrases. ***Evaluate*** the success of this activity by listening for the use of collaboration phrases as well as listening to groups ***Explain*** their processes to see which groups are closest to the process of disk method.

Activity 2: Simulator (20 Min)

After groups share their lemon processes, show students the simulator. Using the simulator, rotate the graph. Ask students what shape would form when we rotate this graph about the x-axis. *A student will say "lemon"*. In table groups, ask students how they think this volume could be calculated and have 2 students share answers. show students using the simulator that if we pretend that the radius is the distance from the graph to the x – axis, and then you rotate the graph, you can form many circles whose areas can be summed together to create the volume. *Students will ask questions here. Draw a diagram on the board if necessary.* Ask students to create an expression with their table groups for disk method and have groups share answers with the class, and how they got there. *Clarify correct formula if necessary*. On the whiteboard, go through the lemon example and calculate the volume of the lemon using the formula.

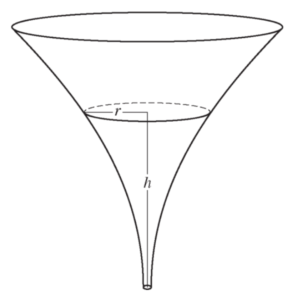
5E Connection

In this part, the teacher ***Elaborates*** on current student knowledge and extends their processes to disk method. The teacher ***Evaluates*** through asking a series a questions during the activity.

Activity 3: My Favorite 'No' (15 Min)

Give each student an index card and project problem 5b from the 2016 AP Calculus AB Exam free response section.

2016 AP® CALCULUS AB FREE-RESPONSE QUESTIONS



5. The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $0 \le h \le 10$. The units of r and h are inches.

(a) Find the average value of the radius of the funnel.

(b) Find the volume of the funnel.

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

Have students attempt to solve individually on the index card. Collect index cards after 5 minutes. Sort through the cards and find an example of a student that got the answer wrong but listed the correct formula and attempted. Write up that solution on the whiteboard to keep anonymity. Ask students what is correct about this student's' work and what is incorrect about this student's work. Correct the student's work on the whiteboard as students are giving suggestions to improve what was incorrect.

5E Connection

In this section, the teacher ***Elaborates*** on previous ideas in order to holistically conclude class.

Methods

Teaching the Lesson

When teaching the lesson, it is important to have the students organized into groups. Mathematics is an extremely collaborative subject and it is important that this collaboration is fostered through a positive learning environment where students are encouraged to work together. When students are finding the volume the lemon, it is important to encourage students to try and connect this part to past calculus concepts. Many students realized that finding this volume was similar to the "Riemann Sum" portion of the class, where students divide the area underneath a graph into rectangles in order to estimate the volume.

In part one of the problem, students share their methods on how they found the volume. It is extremely important that in addition to sharing their actual volumes, they share how they got those volumes. When students share their methods in addition to their answers, it can inspire students how to solve the second part of the problem. It is also critical that the teacher does not highlight a "best" method. This allows students to creatively explore as many methods as possible. This will allow students to come up with unique visualizations that will help answer the research question as well as have students view a variety of visualizations created by other students.

Gathering and Analyzing Data

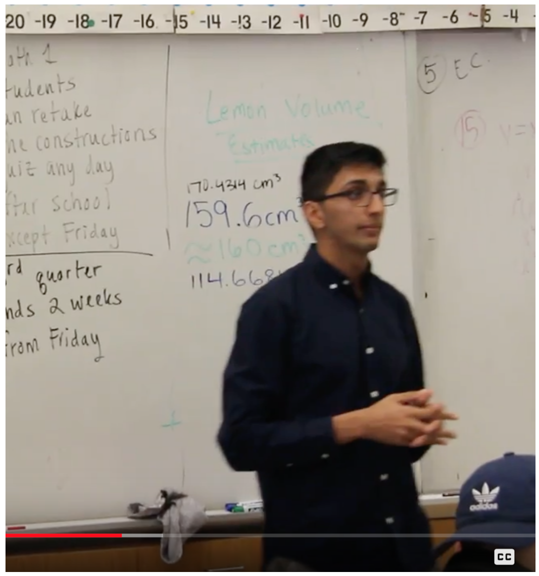
When videoing the lesson, it is important to follow the teacher around the entire lesson and record the interactions that the teacher has with the student visualizations. I viewed these recordings 5 times and transcribed the important interactions in which I take a student visualization and I show it to the class. I analyze this conversation in order to view what impact the visualizations are having on the student learning experience.

In addition to recording the lesson, it is important to take pictures of some of the students' work to highlight what visualizations were used throughout the lesson. Afterwards, it is important to look at these pictures while watching the lesson and see how these visuals were used to help the students learn disk method.

Results

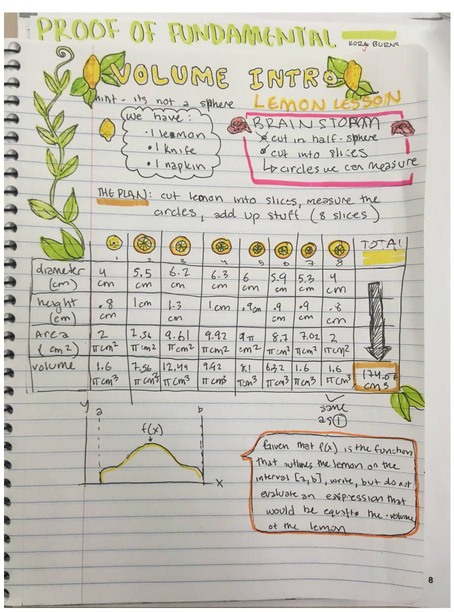
Solving Part 1

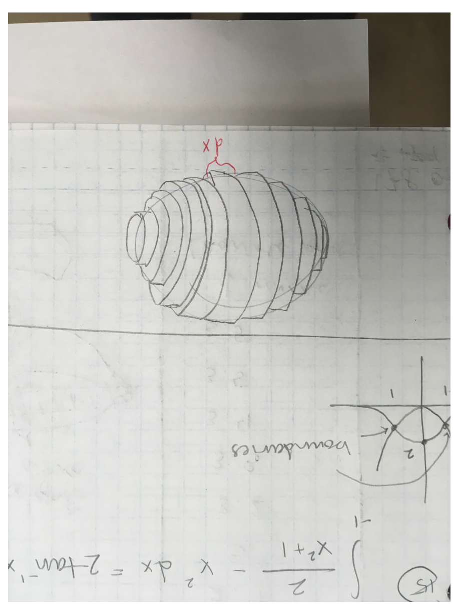
All the student groups tackled the challenge of estimating the volume of the lemon, which solidified its classification as a "low-floor, high-ceiling" problem as mentioned in the introduction. Despite prior student performance in the classroom, all students were willing and were engaged when finding the volume of the lemon. As shown by the below image clip from the video, all student groups had written an estimate on the board.



Visualizations of Part 1

As predicted, a variety of visualizations were used in part 1. However, students geared towards the one-sided or two-sided riemann sum method, in which students cut the lemon into multiple disks and summed the volume of each disk. Here is an example of how students visualized this part of the problem.





As shown by the above diagram, students visualized this part as multiple disks. This implies that students found an inclination to break the larger problem into smaller familiar parts - an important trait to have in a mathematician. The student's work on the left demonstrates how the student wanted to organize the problem as much as possible in order to simplify the problem.

Although disks were the popular option, some students geared toward the common shapes method. Here is an example of an interaction I had with one student group that gravitated towards this method.

Parth: "Do you guys have a plan yet?"
Student 1: "We want to crush it into a cube"
Laughs all around
Student 2: "We are thinking of changing the shape of the lemon itself"
Parth: "Oh I see, you're changing it into a shape that you can take the volume of"
Student 2: "That was the idea"

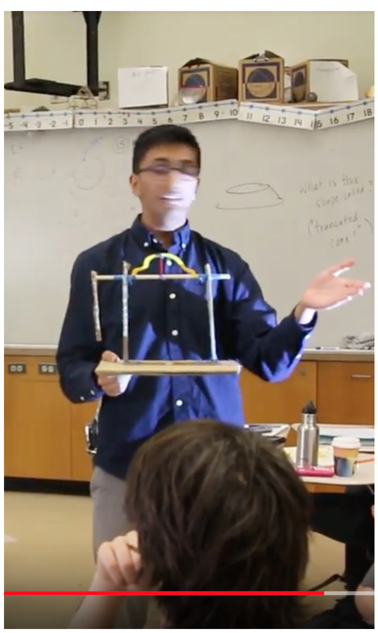
As shown from this interaction, students were also inclined to simplify the problem into something that they are familiar with - a valued trait. I made sure not to tell the students that they were steering away from the “typical” solution because the students were still thinking like mathematicians.

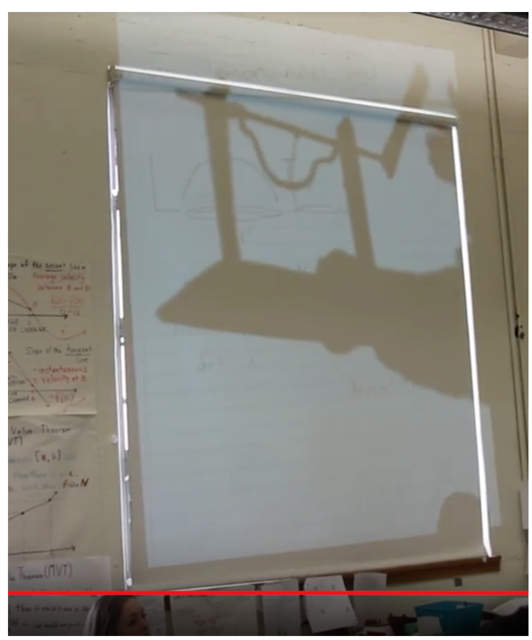
Solving Part 2

All students were able to be engaged with the problem in finding the analytical volume of the rotating lemon graph. However, not all of the solutions were correct. Nevertheless, students were able to demonstrate a mathematical mindset through their visualizations, for mathematics is not about the answer, it is about the process.

Visualizations of Part 2

At the start of this part, I demonstrated my simulation as shown below in order to start the visualization process for students.





For this part, there was a variety of methods used. Many students were building on what they saw in part 1 (a sum of disks). The below transcription shows an example of this.

> **Student**: "So I was saying, if you have slices and all the widths are dx, and then you have different radiuses, I mean I set this into two pieces because it's not necessarily even, so this part might have a different radius here, and the area of this slice is going to be pi - r - squared, so the volume is the area times the width: 'dx'"
> **Parth**: "So what's r?"
> **Student**: "So r is the radius here.."
> **Parth**: "So you're on the right path, but try and do it in terms of the variables in the problem."

As demonstrated by this transcription, this student was building on the slicing of disks in the previous part and was trying to connect it to this part. The student was treating the width of each disk as "dx" and was working towards an integral. This student was symbolic of many students who built upon previous parts which is a great trait to have in an mathematician.

Conclusions

After gathering and analyzing data, it became clear that students use visualizations in order to break a complex problem into smaller parts. I also reaffirmed the importance of low-floor, high-ceiling problems. All of the students were engaged with the problem and felt that they could approach the problem. At the end of the lesson, all of the students could define what disk method was, and the fact that they got to the formula of disk method on their own through visualizations is so much more powerful.

Reflection

If I were to repeat this lesson, I would challenge the groups try and solve the volume of the lemon 2 different ways and have them argue about which way was better. This would allow for more creative methods to be discussed in the class. From this problem, I learned the importance of open-endedness and hands-off teaching in problem because the students learned disk method by themselves. I definitely feel more prepared as a teacher because now I know what types of problems are suitable for the classroom. This lesson has only made me more excited for my career in educational technology.

References

Boaler, Jo. *Mathematical Mindsets: Unleashing Students Potential through Creative Math, Inspiring Messages, and Innovative Teaching*. Jossey-Bass & Pfeiffer Imprints, 2016.